

# Complementary descriptions of shape/phase transitions in atomic nuclei

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## Abstract

Shape/phase transitions in atomic nuclei have first been discovered in the framework of the Interacting Boson Approximation (IBA) model. Critical point symmetries appropriate for nuclei at the transition points have been introduced as special solutions of the Bohr Hamiltonian, stirring the introduction of additional new solutions describing wide ranges of nuclei. The complementarity of the IBA and geometrical approaches will be demonstrated by three examples. First, it will be shown that specific special solutions of the Bohr Hamiltonian correspond to the borders of the critical region of the IBA. Second, it will be demonstrated that the distinct patterns exhibited in different transitional regions by the experimental energy staggering in  $\gamma$ -bands can be reproduced both by the IBA and by special solutions of the Bohr Hamiltonian. Third, a first attempt to obtain a IBA SU(3) level scheme from a special solution of the Bohr Hamiltonian will be presented.

## 1 Introduction

Atomic nuclei are known to exhibit changes of their energy levels and electromagnetic transition rates among them when the number of protons and/or neutrons is modified, resulting in shape phase transitions from one kind of collective behaviour to another. These transitions are not phase transitions of the usual thermodynamic type. They are quantum phase transitions [1] (initially called ground state phase transitions [2]), occurring in Hamiltonians of the type  $H = c(H_1 + gH_2)$ , where  $c$  is a scale factor,  $g$  is the control parameter, and  $H_1$ ,  $H_2$  describe two different phases of the system. The expectation value of a suitably chosen operator, characterizing the state of the system, is used as the order parameter.

In the framework of the Interacting Boson Approximation (IBA) model [3], which describes nuclear structure of even-even nuclei within the U(6) symmetry, possessing the U(5), SU(3), and O(6) limiting dynamical symmetries, appropriate for vibrational, axially deformed, and  $\gamma$ -unstable nuclei respectively, shape phase transitions have been studied 25 years ago [2] using the classical limit of the model [4, 5, 6, 7], pointing out that there is (in the usual Ehrenfest classification) a second order shape phase transition between U(5) and O(6), a first order shape phase transition between U(5) and SU(3), and no shape phase

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transition between  $O(6)$  and  $SU(3)$ . It is instructive to place [1] these shape phase transitions on the symmetry triangle of the IBM [8], at the three corners of which the three limiting symmetries of the IBM appear.

More recently it has been realized [9, 10] that the properties of nuclei lying at the critical point of a shape phase transition can be described by appropriate special solutions of the Bohr Hamiltonian [11], labelled as critical point symmetries. The  $E(5)$  critical point symmetry [9] has been found to correspond to the second order critical point between  $U(5)$  and  $O(6)$ , while the  $X(5)$  critical point symmetry [10] has been found to correspond to the first order transition between  $U(5)$  and  $SU(3)$ .

The introduction of the critical point symmetries  $E(5)$  [9] and  $X(5)$  [10] has triggered much work on special solutions of the Bohr Hamiltonian, corresponding to different physical situations [12, 13]. Several of these solutions will be mentioned below, where the complementarity of the IBA and geometrical approaches will be demonstrated by three examples, given in Sections 2, 3, and 4.

## 2 Connecting the $X(5)$ - $\beta^2$ , $X(5)$ - $\beta^4$ , and $X(3)$ models to the shape/phase transition region of the Interacting Boson Model

As already mentioned in the introduction, shape/phase transitions in atomic nuclei were first investigated [2] within the Interacting Boson Approximation (IBA) model [3] by constructing the classical limit of the model, using the coherent state formalism [4, 5, 6, 7]. Using this method it was shown [2, 6] that the shape/phase transition between the  $U(5)$  [spherical, vibrational] and  $SU(3)$  [prolate axially symmetric deformed, rotational] limiting symmetries is of first order, while the transition between the  $U(5)$  and  $O(6)$  ( $\gamma$ -unstable) limiting symmetries is of second order. Furthermore, the region of phase coexistence within the symmetry triangle [8] of the IBA has been studied [14, 15, 16] and its borders have been determined [17, 18], while a similar structural triangle for the geometric collective model has been constructed [19].

Recently, special solutions of the Bohr Hamiltonian, called critical point symmetries [9, 10], describing nuclei at the points of shape/phase transitions between different limiting symmetries, have attracted considerable attention, since they lead to parameter independent (up to overall scale factors) predictions which are found to be in good agreement with experiment [20, 21, 22, 23]. The  $X(5)$  critical point symmetry [10], was developed to describe analytically the structure of nuclei at the critical point of the transition from vibrational [ $U(5)$ ] to prolate axially symmetric [ $SU(3)$ ] shapes. The solution involves a five-dimensional infinite square well potential in the  $\beta$  collective variable and a harmonic oscillator potential in the  $\gamma$  variable.

The success of the  $X(5)$  model in describing the properties of some nuclei with parameter free (except for scale) predictions has led to considerable interest in such simple models to describe transitional nuclei. Since its development, numerous extensions involving either no free parameters or a single free parameter have been developed. Those approaches which involve a single parameter include replacing the infinite square well potential with a sloped well potential [24], exact decoupling of the  $\beta$  and  $\gamma$  degrees of freedom [25], and displacement

of the infinite square well potential, or the confined  $\beta$ -soft model [26]. Parameter free variants of the X(5) model include the X(5)- $\beta^2$  and X(5)- $\beta^4$  models [27], in which the infinite square well potential is replaced by a  $\beta^2$  and a  $\beta^4$  potential respectively, as well as the X(3) model [28], in which the  $\gamma$  degree of freedom is frozen to  $\gamma = 0$ , resulting in a three-dimensional Hamiltonian, in which an infinite square well potential in  $\beta$  is used.

It is certainly of interest to examine the extent to which the parameter free (up to overall scale factors) predictions of the various critical point symmetries and related models, built within the geometric collective model, are related to the shape/phase transition region of the IBA. It has already been found [29] that the X(5) predictions cannot be exactly reproduced by any point in the two-parameter space of the IBA, while best agreement is obtained for parameters corresponding to a point close to, but outside the shape/phase transition region of the IBA.

In a recent paper [30] the parameter independent (up to overall scale factors) predictions of the X(5)- $\beta^2$ , X(5)- $\beta^4$ , and X(3) models, which are variants of the X(5) critical point symmetry developed within the framework of the geometric collective model, are compared to the results of two-parameter Interacting Boson Approximation (IBA) model calculations, with the aim of establishing a connection between these two approaches. It turns out that both X(3) and X(5)- $\beta^2$  lie close to the U(5)-SU(3) leg of the IBA symmetry triangle and within the narrow shape/phase transition region of the IBA. In particular, X(3) lies close to  $\zeta_{\text{crit}}$  [18], the left border of the shaded shape/phase transition region of the IBA, corresponding to IBA total energy curves with two equal minima, while X(5)- $\beta^2$  lies near the right border of the shape/phase transition region,  $\zeta^{**}$  [17], corresponding to IBA total energy curves with a single deformed minimum. A set of neighboring even-even nuclei exhibiting the X(3), X(5)- $\beta^2$ , and X(5)- $\beta^4$  behaviors have been identified ( $^{172}\text{Os}$ - $^{174}\text{Os}$ - $^{176}\text{Os}$ ) [30]. Additional examples for X(3), X(5)- $\beta^2$ , and X(5)- $\beta^4$  are found in  $^{186}\text{Pt}$ ,  $^{146}\text{Ce}$ , and  $^{158}\text{Er}$ , respectively [30]. The level of agreement of these parameter free, geometrical models with these candidate nuclei is found to be similar to the predictions of the two-parameter IBA calculations.

It is intriguing that the X(3) model, which corresponds to an exactly separable  $\gamma$ -rigid (with  $\gamma = 0$ ) solution of the Bohr collective Hamiltonian, is found to be related to the IBA results at  $\zeta_{\text{crit}}$ , which corresponds to the critical case of two degenerate minima in the IBA total energy curve [18], approximated by an infinite square well potential in the model. It is also remarkable that the X(5)- $\beta^2$  model, which uses the same approximate separation of variables as the X(5) critical point symmetry, is found to correspond to the right border ( $\zeta^{**}$ ) of the shape/phase transition region [17], related to the onset of total energy curves with a single deformed minimum, comparable in shape with the  $\beta^2$  potential used in the model in the presence of a  $L(L+1)/(3\beta^2)$  centrifugal term [25].

Comparisons in the same spirit of the parameter independent predictions of the E(5) critical point symmetry [9] and related E(5)- $\beta^{2n}$  models [31, 32], as well as of the related to triaxial shapes Z(5) [33] and Z(4) [34] models, to IBA calculations and possible placement of these models on the IBA-1 symmetry triangle (or the IBA-2 phase diagram polyhedron [35, 36, 37]) can be illuminating and should be pursued.

It should be noticed that Ref. [30] has been focused on boson numbers equal or close to 10, to which many nuclei correspond. A different but interesting question is to examine if there is any connection between the X(3), X(5)- $\beta^2$ , and/or X(5)- $\beta^4$  models and the IBA for large boson numbers. This is particularly interesting especially since it has been

established (initially for  $N = 1,000$  [31], recently corroborated for  $N = 10,000$  [38]) that the IBA critical point of the U(5)-O(6) transition for large  $N$  corresponds to the E(5)- $\beta^4$  model, i.e. to the E(5) model employing a  $\beta^4$  potential in the place of the infinite well potential [31, 32].

### 3 Staggering in $\gamma$ bands and the transition between different symmetries of nuclear structure

The importance of the  $\gamma$  degree of freedom in nuclei with static quadrupole deformation has been known for decades starting with the work of Wilets and Jean [39], and Davydov, Filippov, and Chaban [40, 41]. Since the bandhead of the quasi- $\gamma$  band is not fixed in most geometrical models, we investigate instead, the parameter free predictions of the spacings within the quasi- $\gamma$  band. For this purpose, we use the odd-even staggering in gamma bands [42]

$$S(J) = \frac{[E(J_\gamma^+) - E((J-1)_\gamma^+)] - [E((J-1)_\gamma^+) - E((J-2)_\gamma^+)]}{E(2_1^+)},$$

which measures the displacement of the  $(J-1)_\gamma^+$  level relative to the average of its neighbors,  $J_\gamma^+$  and  $(J-2)_\gamma^+$ , normalized to the energy of the first excited state of the ground state band,  $2_1^+$ . Since  $S(J)$  is a (discrete) derivative, it is very sensitive to structural changes. For example, the energy levels of the  $\gamma$  band in a  $\gamma$ -independent potential [39] cluster as  $(2_\gamma^+)$ ,  $(3_\gamma^+, 4_\gamma^+)$ ,  $\dots$ , opposite to the rigid triaxial rotor  $(2_\gamma^+, 3_\gamma^+)$ ,  $(4_\gamma^+, 5_\gamma^+)$ ,  $\dots$  clustering pattern [40, 41].

In a recent paper [43], three categories of transitional regions have been considered:

1) The  $\gamma$ -soft region between the vibrator and a deformed  $\gamma$ -soft structure where the potential is  $\gamma$ -independent. This corresponds to the U(5) to O(6) transition in the language of the Interacting Boson Approximation (IBA) [3]. This is the region containing the critical point symmetry E(5) [9], as well as the second-order phase transition in the IBA between U(5) and O(6) [2, 6]. The Xe, Ba, and Ce series of isotopes provide good manifestations of this region [3, 43]. It is seen that  $S(J)$  exhibits strong staggering with minima at even  $J$  and maxima at odd  $J$ . This behavior is reproduced in the geometrical framework by the chain of parameter-independent models formed by E(5) [9], in which an infinite well  $u(\beta)$  potential is used, and the E(5)- $\beta^{2n}$  ( $n = 1, 2, 3, 4$ ) models [31, 32], in which  $u(\beta) = \beta^{2n}/2$ . In all cases the potential is  $\gamma$ -independent. In the IBA it corresponds to  $\chi = 0$  and increasing  $\zeta$  [15, 17].

2) The axially  $\gamma$ -rigid region between the vibrator and the axially symmetric rotor, characterized by a harmonic oscillator in  $\gamma$  with the minimum in  $\gamma$  close to zero. This is the U(5) to SU(3) transition region of the IBA [3], in which a first-order phase transition occurs [2, 6]. This is also the region where the critical point symmetry X(5) [10] is found. Several Sm, Gd, Dy, Er, U, and Fm isotopes exhibit this behavior [3, 43]. It is seen that  $S(J)$  exhibits weak staggering with minima at even  $J$  and maxima at odd  $J$ . This behavior is reproduced in the geometrical framework by the chain of parameter-independent models formed by X(5) [10], in which an infinite well  $u(\beta)$  potential is used, and the X(5)- $\beta^{2n}$  ( $n = 1, 2, 3, 4$ ) models [27], in which  $u(\beta) = \beta^{2n}/2$ , with a potential of the form  $u(\beta) + v(\gamma)$

used in all cases, as well as by the exactly separable ES-X(5) and ES-X(5)- $\beta^2$  models [44], in which potentials of the form  $u(\beta) + v(\gamma)/\beta^2$  are used. In all cases  $v(\gamma)$  is a steep harmonic oscillator potential centered at  $\gamma = 0^\circ$ . In the IBA this region corresponds to  $\chi = -1.32$  and increasing  $\zeta$  [15, 17].

3) The triaxial  $\gamma$ -rigid region between the vibrator and the rigid triaxial rotator [40, 41], characterized by fixed  $\gamma$ -values between  $0^\circ$  and  $30^\circ$ , which has no analog in the framework of IBA-1 [3]. Few nuclei ( $^{112}\text{Ru}$ ,  $^{170}\text{Er}$ ,  $^{192}\text{Os}$ ,  $^{192}\text{Pt}$ ,  $^{232}\text{Th}$ ) are known to show this behavior [43], in which  $S(J)$  exhibits strong staggering with minima at odd  $J$  and maxima at even  $J$ . This behavior is reproduced in the geometrical framework by the chain of parameter-independent models formed by Z(5) [33], Z(5)- $\beta^2$ , Z(4) [34], and Z(4)- $\beta^2$ , which are special solutions of the Bohr Hamiltonian for  $\gamma = 30^\circ$  in five and four dimensions respectively.

Having discussed the dependence of  $S(J)$  on angular momentum, it is interesting to focus attention on  $S(4)$ , the relative displacement of the  $3_\gamma^+$  state relative to the average of its neighbors,  $2_\gamma^+$  and  $4_\gamma^+$ , normalized to the energy of the  $2_1^+$  state, and discuss its variation as a function of changing structure, using as a structure indicator the ratio  $R_{4/2} = E(4_1^+)/E(2_1^+)$ . The following results are found [43]:

i) In the U(5)-O(6) region the IBA predicts  $S(4)$  decreasing with increasing  $R_{4/2}$ . The same prediction is made by the geometrical models mentioned in 1), completed with O(5)-CBS [45], which corresponds to E(5) with an infinite well potential displaced from the origin. The Xe and Ba chains of isotopes provide an experimental manifestation of this behavior.

ii) In the O(6)-SU(3) region the IBA predicts  $S(4)$  increasing with increasing  $R_{4/2}$ . The same prediction is made by the geometrical models mentioned in 3).

iii) In the U(5)-SU(3) region the IBA predicts  $S(4)$  decreasing with increasing  $R_{4/2}$  for low values of  $R_{4/2}$ , then abruptly jumping to a value close to zero, close to which it remains with further increase of  $R_{4/2}$ . The geometrical models mentioned in 2) reproduce the same behavior (slight increase of  $S(4)$  with increasing  $R_{4/2}$ ) beyond the abrupt change, a situation corroborated by experimental evidence from the Nd, Sm, Gd, Dy, and Er isotope chains [43]. The abrupt change appears to be related to the first order phase transition [2, 6] occurring in this region. It appears that phase transitional behavior occurs close to where  $S(4)$  crosses zero.

## 4 Occurrence of IBA SU(3) degeneracy in the geometrical model

A long standing problem is the derivation from the Bohr Hamiltonian [11] of a spectrum similar to that of the SU(3) limit of the Interacting Boson Approximation (IBA) model [3]. The main features of the spectrum should be:

a) The energy spacings among the  $2^+$ ,  $4^+$ ,  $6^+$ , ... levels within the ground,  $\beta$  and  $\gamma$  bands should be identical.

b) Furthermore, the  $2^+$ ,  $4^+$ ,  $6^+$ , ... levels of the  $\beta$  and  $\gamma$  bands should be degenerate.

An attempt to solve this problem has been carried out in a recent paper [46], where an exactly separable version of the Bohr Hamiltonian, called ES-D, which uses a potential of the form  $u(\beta) + u(\gamma)/\beta^2$  [39], with a Davidson potential  $\beta^2 + \beta_0^4/\beta^2$  [47, 48, 49] in the

place of  $u(\beta)$ , and a steep harmonic oscillator centered at  $\gamma = 0^\circ$  as  $u(\gamma)$ , is developed. All bands (e.g., ground,  $\beta$  and  $\gamma$ ) in this model are treated on equal footing [50], depending on two parameters, the Davidson parameter  $\beta_0$  and the stiffness  $c$  of the  $\gamma$ -potential. The model is found [46] to be applicable only to well deformed nuclei (with  $R_{4/2} \geq 3.0$ ) due to the  $\beta^2$  denominator in the  $u(\gamma)$  term. Nevertheless, it reproduces very well the bandheads and energy spacings within bands of almost all rare earth and actinide nuclei, with  $R_{4/2} \geq 3.0$ , for which available data exists, as well as most of the  $B(E2)$  transition rates. The most glaring discrepancy concerns  $B(E2)$  values for the  $\beta$  band to ground band which are typically overpredicted by an order of magnitude. The two exceptions where ES-D does not provide a good description of energy spectra are  $^{152}\text{Sm}$  and  $^{154}\text{Gd}$ , which have previously been shown [22, 51] to be well reproduced with the infinite square well potential of the critical point symmetry X(5). Furthermore, the ES-D model provides insights regarding the recently found correlation [52] between the  $\gamma$  stiffness and the  $\gamma$ -bandhead energy.

Concerning the long standing problem of producing a level scheme with IBA SU(3) degeneracies within the framework of the Bohr Hamiltonian, in the framework of the ES-D model the spacings within the ground and  $\beta$  bands are identical, because of the oscillator term in the  $u(\beta)$  potential. It is therefore enough to examine the conditions under which the  $2^+$ ,  $4^+$ ,  $6^+$ , ... levels of the  $\beta$  and  $\gamma$  bands are degenerate. One is then led [46] to minimize the rms deviation between the even levels of the  $\beta_1$  and  $\gamma_1$  bands for fixed value of the normalized  $\beta_1$ -bandhead,  $R_{0/2} = E(0_\beta^+)/E(2_1^+)$ . Numerical results indicate that a reasonable degree of degeneracy is obtained for  $L_{\max} = 10$  and  $R_{0/2} \geq 15$ , which is of physical interest, since experimental  $R_{0/2}$  values extend up to 27. In the case of  $^{232}\text{Th}$ , in which  $R_{0/2} = 14.8$ , one can see that the above mentioned experimental rms deviation between the even levels of the  $\beta_1$  and  $\gamma_1$  bands is  $\sigma_{\beta,\gamma}^{th}(L_{\max} = 10) = 1.142$ , while the corresponding theoretical quantity is  $\sigma_{\beta,\gamma}^{exp}(L_{\max} = 10) = 0.593$  [46]. Therefore, although the overall fit is quite good, the degree of degeneracy obtained from theory is less than the one indicated by experiment. One could conclude that the ES-D model does contain parameter pairs which correspond to approximate degeneracy of the low lying even levels of the  $\beta_1$  and  $\gamma_1$  bands, while at the same time the spacings within the  $\beta_1$  band are identical to the spacings within the ground band, but the problem of reproducing a SU(3) spectrum from the Bohr Hamiltonian remains conceptually open.

## 5 Conclusion

The field of shape/phase transitions and critical point symmetries is rapidly expanding. Many additional references can be found in the review articles [12, 13, 53], as well as in Ref. [54].

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